Pontryagin principle with a PDE - a unified approach

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Abstract A Pontryagin principle is obtained for a class of optimal control problems with dynamics described by a partial differential equation. The method, using Karush-Kuhn-Tucker necessary conditions for a mathematical program, is almost identical to that for ordinary differential equations.

Keywords Optimal control, Pontryagin principle, partial differential equation, Karush-Kuhn-Tucker conditions

1. Introduction

Pontryagin's principle has been proved in at least four ways, for an optimal control problem in continuous time with dynamics described by an ordinary differential equation (ODE). One approach [5,6] regards the control problem as a mathematical program, and uses the Karush-Kuhn-Tucker (KKT) necessary conditions as the starting point (though with some different hypotheses) for deriving the Pontryagin theory. There are various results for optimal control when the dynamics is described by a partial differential equation (PDE), often derived (e.g. by Lions and Bensoussan) using variational inequalities, which are generally equivalent to mathematical programs in infinite dimensions. The results in [1,2,3,4,5], and others by the same authors, obtain some versions of Pontryagin's principle, by quite different methods to those used for ODEs.

However, the Pontryagin theory involving a PDE can also be derived from the mathematical programming approach, using the KKT conditions, and replacing the time variable t by a space variable z, say in $\mathbf{R}^2$ or $\mathbf{R}^3$, or by (t, z) combined. Whatever approach is followed requires a good deal of detailed calculation, concerned with choice of function spaces (suitable Sobolev spaces), and proofs of differentiability properties. These details are omitted here (they are adequately treated e.g. in [1,3], since the aim here is to show that a Pontryagin principle readily follows. The results depend indeed on certain differentiability properties, stated in what follows, but only indirectly on how these properties are achieved.

2. Pontryagin for ODE

Consider first an optimal control problem with an ODE:

$$\text{MIN } J(u) := F(x,u) := \int_0^T f(x(t),u(t),t)dt \text{ subject to}$$

$$x(0) = x_0, \dot{x}(t) = m(x(t),u(t),t) \ , \ u(t) \in \Gamma(t)(0 \le t \le T) \tag{1}$$

Here $x(.)$ is the state function, $u(.)$ is the control function, the time interval [0, T] is fixed, f and m are differentiable functions. Other details such as variable horizon T, an endpoint constraint on $x(T)$, and state constraints, can readily be added to the problem. They are omitted here, since the purpose is to show the method. The steps are as follows:

(a) The problem (1) is expressed as a mathematical program:

$$\text{MIN}_{x \in X, u \in U} J(u) := F(x,u) \text{ subject to } Dx = M(x,u), u \in \Gamma, \tag{2}$$

over suitable function spaces X and U; X is chosen so that the differential operator $D := (d/dt)$ is a continuous linear mapping [Note 1] .

(b) Assume temporarily that F and M are differentiable with respect to (x,u). Then necessary KKT conditions for a minimum at $(x,u) = (\bar{x},\bar{u})$ are:

$$F_x(\bar{x},\bar{u}) + \hat{\lambda}(-D + M_x(\bar{x},\bar{u}) = 0 \ , \tag{3}$$

$$(F_u(\bar{x},\bar{u}) + \hat{\lambda} M_u \ (\ \bar{x},\bar{u}))(\Gamma - \bar{u}) \ge 0, \tag{4}$$

with a Lagrange multiplier $\hat{\lambda}$. Represent $\hat{\lambda}$ by a function $\bar{\lambda}(.)$, where

$$(\forall w \in \text{ C[0, T] } < \hat{\lambda}, w >= \int_0^T \bar{\lambda}(t), w(t)dt \,. \tag{5}$$

Define the Hamiltonian

$$h(x(t), u(t), t, \lambda(t)) := f(x(t), u(t), t) + \lambda(t)m(x(t), u(t), t); \tag{6}$$

and

$$H(x, u, \hat{\lambda}) := F(x, u) + \hat{\lambda}M(x, u) = \int_0^T h(x(t), u(t), t, \bar{\lambda}(t))dt. \tag{7}$$

In what follows, differentiability will be assumed only with respect to x, not u, so that (4) is not available. The multiplier $\hat{\lambda}$ remains, satisfying (3), provided that the operator $(-D + M_x(\bar{x}, \bar{u})$ is assumed surjective.
(c) Integrating by parts the $-\hat{\lambda}D$ term in (3) leads to:

$$-D\bar{\lambda} = (F + \hat{\lambda}\bar{M})_x(\bar{x}, \bar{u}), \tag{8}$$

if the integrated part vanishes. Choosing a boundary condition to do this, the adjoint differential equation is obtained:

$$-\dot{\bar{\lambda}}(t) = h_x(\bar{x}(t), \bar{u}(t), t, \lambda(t)) \ , \ \bar{\lambda}(T) = 0. \tag{9}$$

(d) Assume that $Dx = M(x, u)$ defines x as a Lipschitz function of u, and that [Note 2]:

$$F(x, u) - F(\bar{x}, u) = F_x(\bar{x}, \bar{u})(x - \bar{x}) + \mathbf{o}(\|x - \bar{x}\| + \|u - \bar{u}\|), \tag{10}$$

with a similar requirement for M. Then minimality of $(\bar{x}, \bar{u})$, namely that $F(x, u) - F(\bar{x}, \bar{u}) \geq 0$, with (3), leads (see [7, Theorem 7.2.3]) to:

$$H(\bar{x}, u, \hat{\lambda}) - H(\bar{x}, \bar{u}, \hat{\lambda}) = F(x, u) - \text{ F}(\ \bar{x}, \bar{u}) + \mathbf{o}(\|u - \bar{u}\|) \geq \ \mathbf{o}\ (\|u - \bar{u}\|) \ , \tag{11}$$

describing a *quasimin* (see [6]) of $H(\bar{x}, ., \hat{\lambda})$ over $\Gamma(.)$ at $\bar{u}$. (Note that there is no requirement of convexity on $\Gamma(.)$.)

(e) Assuming that $\bar{u}$ is a minimum in terms of the L^1 norm, suppose if possible that

$$h(\bar{x}(t), u(t), t, \lambda(t)) < h(\bar{x}(t), \bar{u}(t), t, \lambda(t))$$

for t in a set of positive measure. Then [Note 3] a set of control functions $\{u_\beta\ (.)\ :\ \beta \geq 0\} \subset \Gamma$ is constructed (see [7, Theorem 7.2.6), for which

$$H(\bar{x}, u, \hat{\lambda}) - H(\bar{x}, \bar{u}, \hat{\lambda}) \leq c\|u - \bar{u}\| \tag{12}$$

for some constant $c > 0$, thus contradicting (11). (A required *chattering property* holds automatically for the considered control constraint.) This has proved Pontryagin's principle, in the following form:
Theorem . Let the control problem (1) reach a local minimum at $(x, u) = (\bar{x}, \bar{u}$ with respect to the L^1-norm for the control u. Assume that the differential equation $Dx - M(x, u)$ determines x as a Lipschitz function of u, that the differentiability property (10) (with respect to x) holds, and that $(-D_M(\bar{x}, \bar{u})$ is surjective, Then necessary conditions for the minimum are that the costate $\lambda(.)$ satisfies the adjoint equation (9), and that $h(\bar{x}(t), ., t, \bar{\lambda}(t))$ is minimized over $\Gamma(t)$ at $\bar{u}(t)$, for almost all t.

3. Pontryagin for an elliptic PDE

Denote by Ω a closed bounded region in $\mathbf{R}^3$ (or $\mathbf{R}^2$), with boundary $\partial\Omega$, and disjoint sets $A_i (i = 1, 2, 3, 4)$ whose union is $\partial\Omega$. The optimal problem considered is:

$$\text{MIN } x(.), u(.)J(u) := \int_\Omega f(x(z), u(z), z)dz \tag{13}$$

subject to

$$(\forall z \in \Omega) Dx(z) = m(x(z), u(z), z) \ , \tag{14}$$

$$(\forall z \in A_1) x(z) = x_0(z) \ ; \ (\forall z \in A_2) \nabla x(z)).n(z) = g_0(z); \tag{15}$$

$$(\forall z \in \Omega) u(z) \in \Gamma(z) \ . \tag{16}$$

Here D is an elliptic linear partial differential operator, such as the Laplacian ∇^2, and $n \equiv n(z)$ denotes the outward-pointing unit normal vector to $\partial\Omega$ at $z \in \partial\Omega$. The constraint on the control $u(z)$ is specified in terms of a given set-valued function $\Gamma(z)$. The precise way in which $x(.)$ satisfies the PDE (15) need not be specified here; instead, some specific properties of the solution will be required. The function spaces must be chosen so that D is a continuous linear mapping. This holds, in particular, for $D = \nabla^2$, with Sobolev spaces, if $x \in W_0^2(\Omega)$ and $u \in W_0^1(\Omega)$. It is further required that (14) determines $x(.)$ as a Lipschitz function of $u(.)$. The boundary $\partial\Omega$ of the region need only be smooth enough that Green's theorem can be applied to it.

The Hamiltonian is:

$$h(x(z), u(z), z, \lambda(z)) := f(x(z), u(z), z) + \lambda(z) m(x(z), u(z), z); \tag{17}$$

The steps of Section 2 are now applied, but replacing $t \in [0, \mathrm{T}]$ by $z \in \Omega$. It is observed that steps (a), (b), (d) and (e) remain valid – they do not depend on $t \in \mathbf{R}$. Step (c) requires a replacement for integration by parts. If $D = \nabla^2$, it is appropriate to use Green's theorem in the form:

$$\int_\Omega [\lambda \nabla^2 x - x \nabla^2 \lambda] dv = \int_{\partial\Omega} [\lambda(\partial x/\partial n) - x(\partial\lambda/\partial n] \ \mathrm{ds} \ , \tag{18}$$

in which dv and ds denote elements of volume and surface. The right side of (17) beomes the *integrated part* ; the origin can be shifted, in the spaces of functions, to move $x(z) = x_0(z)$ to $x(z) = 0$, and a similar replacement for the normal component $(\nabla x(z)).n(z).$; so the contributions to the *integrated part* from A_1 and A_2 vanish already. the remaining contributions vanish if boundary conditions are imposed.

$$\lambda(z) = 0 \text{ on } A_3; \partial\lambda/\partial n = 0 \text{ on } A_4 \text{ (thus } \nabla\lambda(z).n(z) = 0 \text{ on } A_4) \tag{19}$$

Then (3) leads to the adjoint PDE:

$$D^* \lambda(z) = \partial h(x(z), u(z), z; \lambda(z))/\partial x(z), \tag{20}$$

with boundary conditions (19), where D^* denotes the adjoint linear operator to D. Here, with $D = \nabla^2$, (17) shows that $D^* = \nabla^2$ also.

Then (e), with $z \in \Omega$ replacing $t \in [0, \mathrm{T}]$, gives Pontryagin's principle in the form: $h(\bar{x}(z), ., t, \bar{\lambda}(z))$ is minimized over $\Gamma(z)$ at $\bar{u}(z)$, possibly except for a set of z of zero measure.

If f and m happen to be linear in u, and if $\Gamma(z)$ is a polyhdron with vertices p_i (or an interval if $u(z) \in \mathbf{R}$), then Pontryagin's principle may lead to *bang-bang control* , namely $u(z) = p_i$ when $z \in E_i$, for some disjoint sets $E_i \subset \Omega$.

4. Pontryagin for a parabolic PDE

Now consider a control problem with dynamics described by the PDE:

$$\partial x(z,t)/\partial t = c^2 \nabla_z^2 x(z,t) + m(x(z,t), u(z,t), z, t) \tag{21}$$

where ∇_z^2 acts on the variable z. Here t (for the ODE) has been replaced by (t, z) $\in [0, T] \times \Omega$, for a closed bounded region $\Omega \subset \mathbf{R}^3$, and $m(.)$ is a forcing function. Define the linear differential operator $D := (\partial/\partial t) - c^2 \nabla_z^2$. The function spaces must be chosen so that D is a continuous linear mapping. Define $A_i \subset \partial\Omega$ as in section 3. The optimal control problem now becomes (with a certain choice of boundary conditions):

$$MIN_{x(.),u(.)}\, J(u) := \int_0^T \int_\Omega f(x(z,t), u(z,t), z, t) dt dz \tag{22}$$

subject to

$$(\forall z \in \Omega) Dx(z,t) = m(x(z,t), u(z,t), t, z)\ , \tag{23}$$

$$(\forall z \in A_1)(\forall t \in [0,T]) x(z,t) = x_0(z,t)\ ;\ (\forall z \in A_2) \nabla x(z,t)).n(z) = g_0(z,t); \tag{24}$$

$$(\forall z \in \partial\Omega) x(z,0) = b_0(z) \tag{25}$$

$$(\forall z \in \Omega) u(z,t) \in \Gamma(z,t)\ . \tag{26}$$

Then steps (a), (b), (d), (e) proceed as in section 3, for an elliptic PDE. The Hamiltonian is:

$$h(x(z,t), u(z,t), z, t, \lambda(z,t)) :=$$

$$f(x(z,t), u(z,t), z, t) + \lambda(z,t) m(x(z,t), u(z,t), z, t); \tag{27}$$

Step (c) (integration by parts) is replaced by the following (denoting I := $[0,T]$ and $\theta := (\partial/\partial t)$) :

$$-\hat{\lambda} Dx = -\int_I \int_\partial \lambda(z,t) Dx(z,t) dz dt = -\int_I \int_\Omega \lambda(z,t)[\theta - \nabla_z^2] x(z,t) dz dt$$

$$= \int_\Omega \mathrm{dz} \int_I (\theta \lambda(z,t)) x(z,t) dt + \int_I \int_\Omega (\nabla_z^2 \lambda(z,t)) x(z,t) dz, \tag{28}$$

applying integration by parts to θ and Green's theorem to ∇_z^2, provided that the "integrated parts" vanish. Since $x(z,t)$ is given for $t = 0$, and for $z \in A_1 \cup A_2 \subset \partial\Omega$, it suffices if

$$(\forall z) \lambda(z,T) = 0, \tag{29}$$

so that $\int_\Omega [\lambda(z,t) x(x,t)]_0^T = 0$, and if

$$(\forall t \in [0,T]) \lambda(z,t) = 0 \text{ on } A_3; \nabla x(z,t.n(z,t) = 0 \text{ on } A_4\ . \tag{30}$$

With these boundary conditions, the adjoint PDE becomes

$$-(\partial/\partial t) \lambda(z,t) = c^2 \nabla_z^2 \lambda(z,t). \tag{31}$$

Then (e), with (z, t) $\in \Omega\times$ I replacing $t \in$ [0, T], gives Pontryagin's principle in the form:

$$h(\bar{x}(z,t), ., z, t, \bar{\lambda}(z,t)) \text{ is minimized over } \Gamma(z) \text{ at } \bar{u}(z,t),$$

possibly except for a set of (z,t) of zero measure.

Concerning bang-bang control, a similar remark to that in section 3 applies here also.

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6. References

1. E. Casas, Boundary control problems for quasi-linear elliptic equations : a Pontryagin's principle, Appl. Math. Optim. **33** (1996), 265-291.

2. E. Casas, Pontryagin's principle for state-constrained boundary control problems of semilinear parabolic equations, SIAM J. Control Opt. **35** , no. 4, (1997), 1297-1327.

3. E. Casas, F. Tröltsch and A. Unger, Second order sufficient optimality condition for a nonlinear elliptic boundary control problem, J. for Analysis and its Applications **15** (1996), 687-707.

4. E. Casas and F. Tröltsch, Second order necessary optimality conditions for some state-constrained control problems of semilinear elliptic equations, Technische Universität Chemnitz, Germany, Preprint (1997).

5. E. Casas, J.-P. Raymond and H. Zidani, Pontryasgtin's principle for local solutions of control problems with mixed control-state constraints, SIAM JOurnal on Control and Optimization 39(4), (1998), 1182-1203.

6. B. D. Craven, *Mathematical Programming and Control Theory* , Chapman & Hall, London (1978)

7. B. D. Craven, *Control and Optimization* , Chapman & Hall, London (1995)

6. Appendix

Note 1 . The linear mapping D is continuous if $x(.)$ is given a *graph norm* :

$$\|x\| := \|x\|^* + \|Dx\|^*,$$

where $\|x\|^*$denotes a given norm, such as $\|x\|_\infty$ or $\|x\|_2$.

Note 2 . It follows from Gronwall's inequality that the mapping from u (with L_1 norm) to x (with $Ldn6\infty$ or L_2 norm) is Lipschitz if $m(.)$ satisfies a Lipschitz condition. The differentiability property (10) replaces the usual $F_x(\bar{x}, \bar{u})$ by $F_x(\bar{x}, u)$. This holds (using the first mean value theorem) if f and m have bounded second derivatives.

Similar results are conjectured for the case of partial differential equations.

Note 3 The construction depends on the (local) minimum being reached wheh u has the L_1 norm, and on the constraint $(\forall z)u(z) \in \Gamma(z)$ having the *chattering property* , that if u and v are feasible controls, then w is a feasible control, defining $w(z) = u(z)$ for $z \in \Omega_1 \subset \Omega$ and $w(z) = v(z)$ for $z \in \Omega \backslash \Omega_1$. For section 4, substitute (z, t) for z here.