

Optimization with generalized invexity

B. D. Craven
University of Melbourne

Abstract Constrained optimization is studied, with nonsmooth (Lipschitz) functions in abstract spaces and cone-constraints. Some more general Lagrangian necessary conditions are obtained, using strict minimum and approximation methods. These conditions are sufficient for a minimum under generalized invex assumptions. A characterization is obtained for generalized invexity, generalizing a known result for differentiable functions. Generalized invexity happens exactly when the generalized Wolfe and Lagrangian dual problems coincide.

Classification : 90C26, 90C48.

Key Words : Constrained optimization, weak minimum, invex, Lipschitz, Lagrangian conditions, infinite dimensions, strict minimum, vector dual, Wolfe dual, Lagrangian dual.

1. Introduction

Consider a constrained optimization problem with nonsmooth (Lipschitz) objective and constraint functions in abstract spaces, and cone-constraints. The objective function may be scalar or vector. Lagrangian necessary conditions for a minimum are well known, under various assumptions. Here some more general necessary conditions are obtained, using an approximation method when the minimum is strict. Vector duality results follow, under *invexity* assumptions.

Many authors have obtained duality results, assuming *invexity*, but have left open the question of when *invexity* occurs. For the case of differentiable functions and finitely many constraints. Craven (2002) gave a characterization of *invexity*, thus describing this property without needing to specify the *scale function* η occurring in its definition. Craven (2005) extended this result to abstract spaces, relevant e.g. to optimal control. This characterization is now extended to *generalized invexity* (for which see Craven, 1986) for nonsmooth (Lipschitz) functions in abstract spaces. A consequence is that the combination of objective and constraint functions is invex exactly when suitable generalizations of the Wolfe dual problem and the Lagrangian dual problem coincide.

2. Formulation

Consider the optimization problem

$$\text{WMIN } F(z) \text{ subject to } -G(x) \in S, \quad (P)$$

in which X and Y are Banach spaces, $Z = \mathbf{R}^r$, $F : X \rightarrow Z$ and $G : X \rightarrow Y$ are locally Lipschitz functions, $Q \subset Z$ and $S \subset Y$ are closed convex cones with interior, and WMIN means *weak minimum* with respect to Q , thus p is a weak minimum of (P) if

$$-G(p) \in S \text{ and } F(z) - F(p) \in W := Z \setminus (-\text{int}Q)$$

whenever $-G(z) \in S$ and z is in a neighbourhood of p . The minimum is *global* if the neighbourhood is all of X .

If F and G are Fréchet-differentiable functions, then necessary Karush-Kuhn-Tucker (KKT) conditions for a weak minimum of (P) at p (where $g(p) \in S$) are:

$$(\exists 0 \neq \tau \in Q^+, \lambda \in S^+) \quad \tau F'(p) + \lambda G'(p) = 0, \lambda G(p) = 0, \quad (K1)$$

where S^+ is the dual cone of S (the set of dual vectors s^+ for which $s^+(S) \subset \mathbf{R}_+$), and Q^+ is the dual cone of Q . If the function $\Phi := (F, G)$ is invex at p with respect to the cone $K := Q \times S$ for some *scale function* η , thus if

$$(\forall z \in X) \quad \Phi(z) - \Phi(p) \in K + \Phi'(p)\eta(x, p), \quad (Ix1)$$

then (K1) is also sufficient for a weak minimum at p .

Instead of differentiable, suppose that F and G are *locally Lipschitz*, thus

$$(\exists \kappa < \infty)(\forall u, v) \in N \quad \|F(u) - F(v)\| \leq \kappa \|u - v\|,$$

where \mathbf{N} is a neighbourhood of p , and similarly for G . Various special cases are known where necessary conditions for a weak minimum are:

$$(\exists 0 \neq \tau \in Q^+, \lambda \in S^+) \quad 0 \in \partial(\tau F + \lambda G)(p), \lambda G(p) = 0, \quad (K2)$$

in which ∂ denotes Clarke subdifferential, or the weaker condition:

$$(\exists 0 \neq \tau \in Q^+, \lambda \in S^+) \quad 0 \in \partial(\tau F)(p) + \partial(\lambda G)(p), \lambda G(p) = 0. \quad (K3)$$

In particular, Clarke (1983) proves (K2) when X and Y have finite dimensions, $Z = \mathbf{R}$, $S = \mathbf{R}_+^m$. Some constraint qualification is required in order to deduce $\tau \neq 0$. Craven (1986) proved (K2) when X and Y have finite dimensions and $Z = \mathbf{R}$, but with a general closed convex cone S .

The proof was based on smoothing the Lipschitz functions, which requires X of finite dimension; however, the assumption of finite dim (Y) was not actually used.

The following questions arise. When does a version of ((K2) or (K3) hold for (P) with Lipschitz functions, when X and Y have infinite dimensions, and S is a general convex cone with interior, not necessarily a polyhedral cone? When can $\partial(\lambda G)(p)$ be replaced by $\lambda \partial^\# G(p)$, when $\partial^\# G(p)$ denotes some version of *generalized Jacobian*? In finite dimensions, Clarke (1983) defines the *generalized Jacobian* as the convex hull of the set of gradients $G'(x)$ at differentiable points $x \rightarrow p$. This requires the differentiable points to be dense in a neighbourhood of p , which follows from Rademacher's theorem when $\dim(X)$ is finite. When is some analog of generalized Jacobian available in infinite dimensions?

Moreover, what version of *invex* is applicable to the nonsmooth problem? In finite dimensions, $G : X \rightarrow Y$ may be defined as *invex* at p (with respect to cone S) if:

$$(\forall x) \quad G(x) - G(p) \in S + \langle \partial^\# F(p), \eta(x, p) \rangle := S + \bigcup_{v \in \partial^\# F(p)} \langle v, \eta(x, p) \rangle \quad (Ix2)$$

(S -convex is the case when $\eta(x, p) = x - p$.) If $e \in \text{int } S$, then $S_1^+ := \{s^+ \in S^+ : \langle s^+, e \rangle = 1\}$ is a *base* for S^+ (thus each element of S^+ has the form αb for some $\alpha \geq 0$ and $b \in S_1^+$), where S_1^+ is convex and weak* compact, by Peressini (1967), Proposition 4.8. Denote by S_g^+ the set of extreme points of S_1^+ . Then (Ix2) is equivalent to the following *generalized invex* (Craven 1988):

$$(\forall x)(\forall b \in S_g^+) \quad bG(x) - bG(p) \geq \partial bG(p)\eta(x, p). \quad (Ix3)$$

The following Alternative Theorem from Glover (1982) is required, generalizing Motzkin's Alternative Theorem.

Theorem 1 (Glover) Let $\varphi : X \rightarrow Y$ be S -sublinear (S -convex and positively homogeneous) and continuous (or weakly Q^* -lsc), and let $\psi : X \rightarrow \mathbf{R}$ be \mathbf{R}_+ -sublinear. Then exactly one holds of

- (i) $(\exists x \in X) \quad -\psi(x) > 0, -\varphi(x) \in S$, and
- (ii) $0 \in \text{cl} [\partial\psi(0) + \bigcup_{s \in Q^+} \partial(s\varphi)(0)]$.

3. Approximation when the minimum is strict

With F, G, Q, S as in (P), and (F, G) invex (Ix2) at p , with cone $Q \times S$, but locally Lipschitz (not always differentiable), does a weak minimum of (P) at p imply a minimum of τF , for some multiplier τ ? This follows from (K1), for a differentiable problem. It also follows from Craven (1986) when X and Y have finite dimensions. But the general question has been open. It can be approached by approximating an infinite-dimensional problem by a finite-dimensional problem, assuming a *strict minimum* of the objective.

Assume here (subject to later construction) that suitable (closed convex) $\partial^\# F(p)$ and $\partial^\# G(p)$ exist. From invex at p ,

$$(\forall x)(\exists \eta) \quad (\exists m = m(x) \in \partial^\# F(p), q = q(x) \in Q^+) \quad F(x) - F(p) = q + m\eta;$$

$$(\forall x)(\exists \eta) \quad (\exists n = n(x) \in \partial^\# G(p), s = s(x) \in S^+) \quad F(x) - F(p) = q + m\eta.$$

Then p is a weak minimum of (P)

$$\Leftrightarrow (\text{NOT } \exists x) \quad F(x) - F(p) \in -\text{int } Q, G(x) \in -S,$$

$$\begin{aligned}
&\Leftrightarrow \text{NOT } (\exists \eta, m \in \partial^\# F(p), n \in \partial^\# G(p), q \in Q^+, s \in S^+) \\
&\quad q + m\eta \in -\text{int } Q, G(p) + s + n\eta \in -S \\
&\Rightarrow \text{NOT } (\exists \eta, m, n) \quad m\eta \in -\text{int } Q, G(p) + n\eta \in -S, \\
&\Rightarrow (\exists 0 \neq \tau \in Q^+, \lambda \in S^+) \quad \tau m + \lambda n = 0, \lambda G(p) = 0 \\
&\quad \text{by Motzkin, under the a } \textit{closed-cone assumption} \\
&\quad \text{(see Craven (1995) that } n^*(S + G(p)) \text{ is weak } * \text{ closed,} \\
&\quad \text{where } n^* \text{ denotes adjoint mapping of } n \\
&\Rightarrow (\forall x, -G(x) \in S) \quad \tau[F(x) - F(p)] \geq (\tau F + \lambda G)(x) - (\tau F + \lambda G)(p) \\
&\quad \geq (\tau m + \lambda n)\eta(x, p) = 0 \quad \text{by the invex assumption.}
\end{aligned}$$

Hence a nonzero multiplier $\tau \in Q^+$ exists, such that $\tau F(x)$ reaches a minimum over $-G(x) \in S$ at p .

It remains to construct suitable *generalized Jacobians* for problems with infinite dimensions. Assume now that (P) reaches a weak minimum at p , and $\dim(X) = \infty$. Assume now that the constrained minimum of $\tau F(\cdot)$ is *strict*, thus for sufficiently small $r > 0$, there exists $\delta(r) > 0$, such that

$$(-G(x) \in S, \|x - p\| = r) \quad \tau F(x) - \tau F(p) \geq \delta(r).$$

Now specialize X to Hilbert space of sequences $x = \{x^1, x^2, x^3, \dots\}$ with norm $\|x\| = (\sum \omega_i (x^i)^2)^{1/2}$.

Denote by X_n the subspace of X , obtained by setting components x_j to 0, for all $j > n$. Denote by (P_n) the truncated problem, obtained from (P) by adjoining the constraint $x \in X_n$. Let $F^n(x) := F(x^1, x^2, \dots, x^n, 0, 0, \dots)$, and similarly for G^n . Assume (by choice of weights ω_i) that

- (i) $f^n(x)$ and $g^n(x)$ are uniformly continuous in $(x, 1/n)$ in a bounded region around $(p, 0)$, and that
- (ii) $f^n(\cdot)$ reaches a minimum on each closed bounded set.

From Craven (1995), the strict minimum, with (i) and (ii), ensure that (P_n) reaches a minimum at a point p^n , where $p^n \rightarrow p$ as $n \rightarrow \infty$, and (KKT) holds with $0 \in \partial(\tau f^n + \lambda^n g^n)(p^n)$. (See Craven (1988) for proof of these (KKT) conditions, assuming $\dim(X^n) < \infty$, The discussion of strict minimum and $p^n \rightarrow p$ in Craven (1995) assumes $\dim(Y) < \infty$, but that hypothesis is not used.)

Now normalize the multiplier vector $(1, \lambda^n)$ to $v^n := \alpha_n(1, \lambda^n)$, with α_n so that $\langle v^n, e \rangle = 1$. By weak * compactness of the base S_b^+ , v^n tends to a weak * subsequence limit $v = (\gamma, \lambda) \neq (0, 0)$ as $n \rightarrow \infty$. Similarly, the elements of $\partial^\# F^n(p^n)$ and $\partial^\# G^n(p^n)$, being weak * bounded, tend to weak * subsequence limits as $n \rightarrow \infty$. These limit sets now define $\partial^\# F(p)$ and $\partial^\# G(p)$.

Assume the *constraint qualification*:

$$0 \in \text{int } \{G(p) + \langle \partial^\# G(p), X \rangle - S\}. \quad (CQ)$$

Hence $0 \in \text{int } [\langle \partial(\lambda G)(p), X \rangle - \mathbf{R}_+]$, which is contradicted if $\lambda = 0$, implying $\gamma \neq 0$.

The following theorem has thus been proved.

Theorem 2 Assume that

- the function (F, G) is locally Lipschitz and invex (Ix2), with $\partial^\# F(p)$ and $\partial^\# G(p)$ defined, as above, by subsequence limits;
- the problem (P), with X a Hilbert space, reaches a weak minimum at p , with multipliers τ, λ ;
- the constraint qualification (CQ) holds.
- $\tau F(\cdot)$ reaches a constrained a strict local minimum at a point p ;
- the closed-cone assumption holds;
- the hypotheses (i) and (ii) hold, relating to truncation of infinite dimensional problems;

Then necessary and sufficient conditions of form (K3) hold for a weak minimum of (P) at p .

4. Vector duality

In this section, assume that (P) reaches a weak minimum at p , and that necessary conditions (K3) hold there. Assume also that F and G are invex (as (Ix2)) at each point, with the same η . Following Craven (1989), with WMAX Φ meaning WMIN $(-\Phi)$, the *modified Wolfe dual* is:

$$\begin{aligned} & \text{WMAX}_{u,V} F(u) + VG(u) \text{ subject to } V(S) \subset Q, \\ & (\exists \sigma \in \partial F(u))(\exists \omega \in \partial G(u)) \quad (\sigma + V\omega)(X) \subset W. \end{aligned} \quad (MWD)$$

The dual variables are $u \in X$ and the continuous linear mapping V .

Theorem 3 Let F and G be invex, with respect to the same scale function η . Let (K3) hold for a point p , with $-G(p) \in S$. Then there hold:

weak duality (WWD): If u is feasible for the given problem, and (u, V) is feasible for the dual problem, then $F(x) - [F(u) + VG(u)] \in W$.

Zero duality gap (ZDG) The objective functions are equal at p , thus:

$$F(p) = F(u) + VG(u).$$

Remark WWD indicates *weak duality* (as for scalar-valued optimization problems) in relation to *weak minimum* and *weak maximum*.

Proof Let x and (u, V) be feasible for the respective problems. Then

$$\begin{aligned} F(x) - [F(u) + VG(u)] &= -VG(x) + [F(x) + VG(x)] - [F(u) + VG(u)] \\ &\in Q + (F + VG)^\circ(u, \eta(x, u)) + Q && \text{by invexity} \\ &\subset Q + (\sigma + V\omega)(X) \subset Q + W \subset W. \end{aligned}$$

So WWD holds; and ZDG follows from (K3).

5. Characterizing invex

A characterization of *invex* was given in Craven (2002) for differentiable functions. Characterizations for locally Lipschitz functions are obtained as follows.

Consider a locally Lipschitz function $H : X \rightarrow Y$, with convex cone $S \subset Y$ and $\text{int } S \neq \emptyset$. For each $b \in \mathbf{A} := S_1^+$, define $\tilde{H}(x) \in C(S_1^+)$ by $\tilde{H}(x)(b) := bH(x)$, and define $\tilde{M}(\eta) \in C(S_1^+)$ by

$$\tilde{M}(\eta)(b) := (bH)^\circ(p; \eta).$$

Then $\tilde{M}(\cdot)$ is P -sublinear, where P is the positive cone (of pointwise nonnegative functions) in $C(S_b^+)$. The elements of the dual space of $C(S_b^+)$ are (regular Borel) *measures* on $C(S_b^+)$ (see e.g. Taylor, 1958, section 7.5); denote the space of such measures by \mathbf{M} . If $\dim(Y) < \infty$, then the base S_b^+ may be replaced by the set S_g^+ of generators, which is then compact. Given $x \in X$, denote $\tilde{c} := \tilde{H}(x) - \tilde{H}(p)$. From (Ix3), H is *invex* at p when η exists (depending on x), satisfying $\tilde{c} - \tilde{M}(\eta) \in P$.

Theorem 4 H is invex at p for some scale function η , if and only if, for $\mu \in \mathbf{M}$,

$$0 \in \mathbf{K}(\mu) \Rightarrow \int_{\underline{\mathbf{f}}_{\mathbf{A}}} \langle \mu(db), b[H(x) - H(p)] \rangle \geq 0.$$

where $\mathbf{A} := S_1^+$ and

$$\mathbf{K}(\mu) := \int_{\mathbf{A}} \langle \mu(db), \partial(bH)(p) \rangle.$$

Proof Invexity holds if and only if (for each x):

$$\begin{aligned} & (\exists \eta) \tilde{c} - \tilde{M}(\eta) \in P \Leftrightarrow (\exists z = t\eta, t > 0) \tilde{M}(z) - \tilde{c}t \in P, t \in \text{int } \mathbf{R}_+ \\ & \Leftrightarrow \text{NOT } (\exists \mu \in \mathbf{M}, 0 \neq \beta \in \mathbf{R}_+) \quad 0 \in \text{cl} \int_{\mathbf{A}} \langle \mu(d\xi), \partial \tilde{M}(0)(\xi) \rangle, \int_{\mathbf{A}} \langle \mu(d\xi), c^+(\xi) \rangle \leq 0 \end{aligned}$$

by Glover's alternative theorem (Theorem 1), integrating over $\mathbf{A} := S_1^+$

$$\Leftrightarrow (\exists \mu \in \mathbf{M}) \quad 0 \in \mathbf{K}(\mu) \Rightarrow \int \langle \mu(db), c^+(b) \rangle \geq 0,$$

where $c^+(b) := b\tilde{c} = b[\tilde{H}(x) - \tilde{H}(p)]$, since from $(bH)^\circ(y) = \sup\{\langle v, y \rangle : v \in \partial(bH)(p)\}$ and $\partial\langle \xi, \cdot \rangle = \xi$, the expression for the subdifferential of a supremum in Clarke (1983, Theorem 2.8.2) shows that $\partial(bH)^\circ(0) = \text{cl co } \partial(bH)(p) = \partial(bF)(p)$. This Clarke theorem requires that $T := \partial(bH)(p)$ is sequentially compact, the mapping $\langle \cdot, y \rangle$ is usc, Lipschitz, closed, and Clarke regular, and that the sup is bounded; then

$$\partial(bH)^\circ(y) = \left\{ \int_T \mu(db)\xi : \mu \in \mathbf{P}(T) \right\} = \xi,$$

where $\mathbf{P}(T)$ denotes probability measures on T .

6. Lagrangian dual and invexity

Consider the problem:

$$\text{MIN } F(x) \text{ subject to } -G(x) \in S, \quad (P2)$$

where $F(\cdot)$ takes real values, and the other symbols are as in (P), and the associated problem:

$$\text{MAX } \Phi(v) \text{ subject to } v \in S^+, \quad \text{where } \Phi(v) := \text{MIN}_u F(u) + vG(u). \quad (D2)$$

Assuming that F is convex, G is S -convex, and a constraint qualification, (D2) is the Lagrangian dual problem to (P2).

If the constraint $-G(x) \in S$ consists of inequalities $G_j(x) \leq 0$ ($j = 1, 2, \dots, m$), denote the problems (P2) and (D2) by (P3) and (D3.) In this case, Theorem 4 characterizes invexity of (F, G) by:

$$[(\gamma, v) \geq (0, 0), 0 \in \gamma\partial F(p) + \sum v_j\partial G_j(p)] \Rightarrow \gamma F(\cdot) + \sum v_j\partial G_j(\cdot) \geq \gamma F(p) + \sum v_j\partial G_j(p). \quad (Ix^*)$$

Theorem 5 Consider the problem (P3). For each point (u, v) satisfying

- (i) $v = (v_1, v_2, \dots) \geq 0$
- (ii) $0 \in \partial F(u) + \sum v_j\partial G_j(u)$,
- (iii) $0 \notin \sum v_j\partial G_j(u)$, when $v \neq 0$,

the function $H := (F, G)$ is invex at (u, v) if and only if:

$$F(\cdot) + \sum_j v_j G_j(\cdot) \geq F(u) + \sum_j v_j G_j(u). \quad (\text{MinLagr})$$

Remarks Hypotheses (i) and (ii) state that (u, v) is a feasible point for the Wolfe dual problem $WD2$.

$$\text{MAX}_{u,v} F(u) + \sum_j v_j G_j(u) \text{ subject to } (\forall j)v_j \geq 0, 0 \in \partial F(u) + \sum_j v_j\partial G_j(u).$$

Hypothesis (iii) is a constraint qualification, related to the Slater constraint qualification.

For given u , note that $0 \in \gamma\partial F(p) + \sum v_j\partial G_j(p)$ is often only satisfied when $(\gamma, v) = \beta(1, \lambda)$ for a unique Lagrange multipliers λ , and positive values of β .

Proof From Theorem 4, H is invex at u . if and only if (Ix^*) holds. Because of the constraint qualification (iii), the case $\gamma = 0$ does not hold in (Ix^*) . Hence, dividing by γ , or equivalently replacing γ by 1 in (Ix^*) , (Ix^*) is here equivalent to $[(i) + (ii)] \Rightarrow (\text{MinLagr})$.

Consider now the problems (P2) and (D2) with $F(\cdot)$ real-valued, but without restriction to a finite set of inequality constraints. Then $\mathbf{K}(\mu)$ in Theorem 4 can be expressed as:

$$\mathbf{K}(\mu) = \gamma\partial F(p) + \partial(vG)(p),$$

with Lagrange multipliers

$$(0, 0) \neq (\gamma, v) := \int_{\mathbf{A}} \mu(db) \quad b \in \mathbf{R}_+ \times S^+$$

Then (Ix^*) is replaced by:

$$(\gamma, v) \in \mathbf{R}_+ \times S^+, \quad 0 \in \gamma \partial F(p) + \partial(vG)(p) \Rightarrow \gamma F(\cdot) + vG(\cdot) \geq \gamma F(p) + vG(p). \quad (Ix^{**})$$

Theorem 6 Consider the problem $(P2)$ with real-valued objective $F(\cdot)$. For each point (u, v) satisfying:

- (i) $v \in S^+$
- (ii) $0 \in \partial F(u) + \partial(vG)(u)$,
- (iii) $0 \notin \partial(vG)(u)$, when $v \in S^+$ and $v \neq 0$,

the function $H := (F, G)$ is invex at (u, v) if and only if:

$$F(\cdot) + vG(\cdot) \geq F(u) + vG(u).. \quad (MinLagr2)$$

Proof From Theorem 4 and the above discussion, H is invex at u if and only if (Ix^{**}) holds. Because of the constraint qualification (iii), the case $\gamma = 0$ does not hold in (Ix^{**}) . Hence, dividing by γ , or equivalently replacing γ by 1 in (Ix^{**}) , it follows that (Ix^{**}) is here equivalent to $[(i) + (ii)] \Rightarrow (MinLagr2)$.

Finally, consider problems $(P2)$ and $(D2)$ with unrestricted constraints, a vector valued objective $F(\cdot)$, and weak minimization.

Theorem 7 Consider the problem $(P2)$ with vectorvalued objective $F(\cdot)$. For each point (u, v, τ) satisfying:

- (i) $v \in S^+$ and $0 \neq \tau \in Q^+$,
- (ii) $0 \in \partial(\tau F)(u) + \partial(vG)(u)$,
- (iii) $0 \notin \partial(vG)(u)$, when $0 \neq v \in S^+$,

the function $H := (F, G)$ is invex at (u, v, τ) if and only if:

$$[\tau F(\cdot) + vG(\cdot)] - [\tau F(u) + vG(u)] \geq 0. \quad (MinLagr3)$$

Proof From Theorem 4, $H(\cdot)$ is invex at (u, v, τ) if and only if:

$$[\tau \in Q_+, v \in S^+, \quad 0 \in \partial(\tau F)(p) + \partial(vG)(p)] \Rightarrow \tau F(\cdot) + vG(\cdot) \geq \tau F(p) + vG(p). \quad (Ix^{**})$$

The case $\tau = 0$ is excluded by hypothesis (iii), so the conclusion follows.

Remarks As in Craven (1989), *MinLagr3* may be expressed as a vector inclusion:

$$[F(\cdot) + VG(\cdot)] - [F(u) + VG(u)] \in Z \setminus (-\text{int}Q),$$

where the linear mapping V satisfies $\tau V = v$, and $V(S) \subset Q$.

Acknowledgement

Thanks are due to two referees for various corrections to this paper.

References

- Clarke, F., 1983. *Optimization and Nonsmooth Analysis*, Wiley, New York.
- Craven, B. D., 1986. Nondifferentiable optimization by smooth approximations, *Optimization* 17, 3-17.
- Craven, B. D., 1989 A modified Wolfe dual for weak vector minimization, *Numerical Functional Analysis and Optimization*, 101, 899-907
- Craven, B. D., 1995. *Control and Optimization*, Chapman & Hall, London.
- Craven, B. D., 2002. Global invexity and duality in mathematical programming, *Asia-Pacific Journal of Operational Research*, 19, 169-175.
- Craven, B. D., 2005. Characterizing invex and related properties. *Generalized Convexity, Generalized Monotonicity, and Applications*, Eberhard, A., Hadjisavvas, N., and Luc, D. T. (Eds.), Springer, New York, 183-191.
- Glover, B. M, 1982 Generalized Farkas lemma and application to quasidifferentiable programming, *Zeitschrift für Operations Research* 46, 124-141.
- Peressini, A. L., 1967. *Ordered Topological Vector Spaces*. Harper & Row, New York.
- Taylor, A., 1958. *Introduction to Functional Analysis*, Wiley, New York.